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Abstract

Victor prefers safety more than Ursula if whenever Ursula prefers some constant to some uncertain act, so does Victor. This paradigm, whose Expected Utility version takes the form of Arrow & Pratt's more risk averse concept under which Victor's utility v and Ursula's utility u must be such that $v(u^{-1}(\cdot))$ is concave, will be studied in the Choquet Uncertainty model, letting μ (ν) be Ursula's (Victor's) capacity.

A necessary and sufficient condition (A) on the pairs (u, μ) and (v, ν) will be presented for *dichotomous* weak increased uncertainty aversion, the preference by Victor of a constant over a dichotomous act whenever such is the preference of Ursula. This condition, pointwise inequality between a function defined in terms of $v(u^{-1}(\cdot))$ and another defined purely in terms of the capacities, preserves the flavor of the "more pessimism than greediness" characterization of monotone risk aversion by Chateauneuf, Cohen & Meilijson in the Rank-dependent Utility Model and its extension by Grant & Quiggin to the Choquet Utility Model.

A sufficient condition (B) in terms of the capacities only, satisfied in particular if $\nu(\cdot) = f(\mu(\cdot))$ for some convex f, will be presented for *simplicity seeking*, the preference by Victor over any act for some dichotomous act that leaves Ursula indifferent. Condition A is thus a characterization of weak increased uncertainty aversion for convex f.

An example will be exhibited disproving the more far reaching conjecture under which the dichotomous case implies the general case.

Keywords and Phrases: Choquet Utility, Greediness, Pessimism, Rank-dependent Utility, Risk aversion, Uncertainty.

JEL Classification Numbers: D81

1 Introduction

Risk aversion depends on the notion of risk applied: a dichotomy averse decision maker (DM) prefers to every dichotomous random variable, its mean. A weakly risk averse DM prefers to every integrable random variable, its mean. A strongly risk averse DM prefers less risk to more risk in the sense of Mean-preserving increase in risk (MPIR; Hardy, Littlewood & Polya [19], Rothschild & Stiglitz [27, 28]). A monotone risk averse DM prefers less risk to more risk in the sense of Quiggin [25] (see also Bickel & Lehmann [1, 2]). The literature contributes a number of in-between notions of risk and risk aversion, such as selective risk aversion (Landsberger & Meilijson [21]) and Location-independent risk (Jewitt [20]).

In the von Neumann & Morgenstern expected utility (EU) setup the DM is fully characterized by a utility function. Dichotomy aversion implies concavity of the utility function, and this implies all other types of risk aversion listed above, rendering all equivalent to each other, well represented by MPIR.

Degree of risk aversion is a more delicate concept in EU: the well accepted Arrow & Pratt characterization of weak risk aversion applies to aversion to monotone risk (even in the class of non-decreasing utility functions, Landsberger & Meilijson [22]) and to Location-independent risk (in the class of concave non-decreasing utility functions, Jewitt [20]) but not to MPIR (Ross [26]).

The rank-dependent expected utility model (RDU; Quiggin [24], Yaari [32]) replaces EU by the more general *RDU* functional as an index of preference between measurable functions (random variables, acts), in terms of a *probability transformation function* f that transforms the distribution function F much in the same way the utility function transforms wealth. The DM is characterized by the pair (utility function on wealth, probability transformation function). RDU is expected utility taken with respect to the transformed distribution function $1 - f(1 - F)$.

In RDU, strong risk aversion is characterized by concavity of the utility function and convexity of the probability transformation function (Chew, Karni & Safra [10]). While monotone risk aversion *with concave utility function* is characterized by plain pessimism (the transformed probability is first-degree dominated by the original probability; Quiggin [24, 25]), it does not require concavity of the utility function and is characterized by "more pessimism than greediness" ($P_f \geq G_u$; Chateauneuf, Cohen & Meilijson [9]), in terms of a scalar index of pessimism ($P_f \geq 1$) derived from the probability transformation function and a scalar index of greediness (or non-concavity $G_u \geq 1$) derived from the utility function. After establishing that $G_u = 1$ if and only if u is concave and $P_f = 1$ for linear f, this characterization is seen to generalize the role played by concavity in EU and in Chew, Karni & Safra's result for RDU.

Since EU and RDU assume a bona-fide distribution, EU and RDU are indices under risk. In the Choquet uncertainty model (CEU) (Schmeidler [29, 30]), the DM is characterized by a utility function on wealth and a Choquet capacity [11], a monotone but not necessarily additive set function, normalized so as to assign value 0 to the empty set and 1 to the whole space. Acts are compared by Choquet integrals of the utility function with respect to the capacity. This setup models uncertainty, wherein the DM doesn't necessarily specify a probability measure on events. In contrast, RDU and EU model probability sophistication, via a well specified probability measure on events.

Under uncertainty, there is no conceivably proper notion of uncertainty aversion, since there is no objective nor subjective probability measure to serve as anchor for the evaluation of an expectation. The closest natural notion we tried, the Shapley value of the coalition game defined by the capacity, proved inadequate.

Earlier attempts by Chateauneuf & Cohen [6] to characterize weak risk aversion produced necessary conditions on the one hand and sufficient conditions (allowing non-concave u) on the other, but no characterization.

The key in Epstein [16], Ghirardato & Marinacci [17] and Grant & Quiggin's [18] approaches has been to extend from risk to uncertainty the notion of *more* risk averse rather than that of risk averse. Grant & Quiggin's (GQ; [18]) lifted comparative pessimism and greediness analysis from RDU to CEU by interpreting a monotone risk averse DM as one that is monotone more risk averse than a risk neutral DM. Victor (with utility function v and capacity ν) is monotone more uncertainty averse than Ursula (u and μ) if Victor always prefers monotone less uncertainty that leaves Ursula indifferent. The GQ relative index of pessimism of ν with respect to μ is roughly defined as the CCM index of pessimism, letting μ take the role of the probability measure and ν that of the transformed probability measure. Similarly, the GQ index of relative concavity of v with respect to u is defined as the CCM index of greediness of $w(\cdot) = v(u^{-1}(\cdot))$. Grant & Quiggin proved that Victor is monotone more uncertainty averse than Ursula if and only if Victor is relatively more pessimistic than greedy with respect to Ursula.

This state of affairs is awkward: we understand the elaborate notions of strong and monotone risk and uncertainty aversion as well as some other stochastic variations, but not yet the most natural, Arrow & Pratt-type elementary notion of weak risk and uncertainty aversion, the preference for safety: Victor is weakly more uncertainty averse than Ursula if whenever Ursula prefers some constant to some uncertain act, so does Victor. It seems easier to study stochastic orders with simple progressive generators that allow analysis to transit from less risk to more risk by well understood simple spreads, than it is to transit from risk to safety in one daring jump.

Monotone risk aversion implies weak risk aversion and monotone more uncertainty aversion implies monotone more weak uncertainty aversion, so the necessary and sufficient conditions for the monotonicity versions are sufficient conditions for the weakness versions. Examples provided in the sequel will show that these are not necessary.

Concentrating on two-valued acts, a characterization will be provided for dichotomous weak risk (and increasing uncertainty) aversion, in terms of a *lower envelope* function LE_w that depends on u and v only (via $w(\cdot) = v(u^{-1}(\cdot))$), and a *capacity transformation function* $f_{\mu,\nu}$ depending on μ and ν only. Both are increasing functions from $[0, 1]$ to itself, that leave the endpoints invariant: $f_{\mu,\nu}(q)$ is the supremum of $\nu(A)$ taken over all A with $\mu(A) \leq q$. $LE_w(x)$ is the infimum over all $a < b$ of $w_{a,b}(x)$, where the "zooms" $w_{a,b}$ of w are defined by $w_{a,b}(x) = \frac{w(a+x(b-a))-w(a)}{w(b)-w(a)}$.

We will prove that Victor is dichotomous weakly more uncertainty averse than Ursula if and only if $f_{\mu,\nu} \leq LE_w$ (pointwise). Furthermore, if $f_{\mu,\nu} \leq g \leq LE_w$ for some convex g (in particular, if either LE_w or $f_{\mu,\nu}$ is convex), Victor is weakly more uncertainty averse than Ursula. This statement holds also under the weaker assumption of separation by *semi-convex* g (a notion to be introduced) as a sufficient condition for *simplicity seeking*, the preference by Victor over every distribution for some dichotomous distribution that leaves Ursula indifferent.

These are functional conditions that postulate pointwise inequality between functions (such as the Arrow-Pratt index or the integral condition for second degree dominance), unlike the scalar condition for monotone more uncertainty aversion above, that postulate a much simpler inequality between two numbers. The conditions are similar in the sense that each compares an object defined in terms of w only (that measures departure from concavity) with another defined in terms of $f_{\mu,\nu}$ only (that measures departure from linearity).

The lower envelope function LE_w can't be any increasing function. However, self-zooming functions ($LE_w \equiv w$) trivially belong to the class. Via a deep connection between this setup and the casino theory of Dubins & Savage ([15], Theorem 4.2.1. and expression 4.2.8), self-zooming functions are precisely those that can be obtained as the optimal probability $U(\cdot)$ of reaching the goal in subfair casinos. The U function of Red & Black casino will be shown to provide a counterexample to the plausible conjecture that dichotomous weak uncertainty aversion implies weak uncertainty aversion. However, the method of proof of the theorem (Dubins [13], Smith [31]) claiming that "there is no advantage in ever placing a bet on more than one hole on any single spin of the roulette wheel", will be applied to show that semi-convex $f_{\mu,\nu}$ satisfy the conjecture.

When the capacity transformation function $f_{\mu,\nu}$ is the identity (i.e., $\nu \equiv \mu$), the statement $f_{\mu,\nu} \leq LE_w$ holds if and only if w is concave. This argument provides a re-derivation of the Arrow-Pratt characterization of weak or increasing weak risk aversion under EU. Thus, the conditions for dichotomous or more general weak uncertainty aversion presented here, lift the Pratt paradigm from EU to CEU.

2 The main results

This paper deals with three concepts: *weak increasing uncertainty aversion* (or increasing preference for safety versus uncertain acts), *dichotomous* weak increasing uncertainty aversion (or increasing preference for safety versus dichotomous acts) and *simplicity-seeking preferences* (or preference over any uncertain act of some dichotomous act).

Dichotomous increasing uncertainty aversion. Let (u, μ) and (v, ν) be the utility functions and Choquet capacities of two CEU DM's Ursula and Victor. Victor is more averse to dichotomous acts versus safety than Ursula if for dichotomous acts X with payoff x_3 on A and x_1 otherwise, the inequality

$$u(x_1) + \mu(A)(u(x_3) - u(x_1)) \leq u(x_2)\ ,\ x_1 < x_2 < x_3 \tag{1}$$

implies

$$v(x_1) + \nu(A)(v(x_3) - v(x_1)) \leq v(x_2) \tag{2}$$

that is, if whenever Ursula prefers a constant x_2 to X, so does Victor. Rephrasing these inequalities using the shorthand $u_i = u(x_i)$ and $w(u(\cdot)) = v(\cdot)$, if $\mu(A) \leq \frac{u_2 - u_1}{u_3 - u_1}$ implies

$\nu(A) \leq \frac{w(u_2)-w(u_1)}{w(u_3)-w(u_1)}$. Define the *capacity transformation function* $f_{\mu,\nu} : [0,1] \to [0,1]$ by

$$f_{\mu,\nu}(q) = \sup_{\{A|\mu(A)\leq q\}} \nu(A) \tag{3}$$

and the *lower envelope function* $LE_w : [0,1] \to [0,1]$ by

$$LE_w(q) = \inf_{\{u_1<u_2<u_3|\frac{u_2-u_1}{u_3-u_1}=q\}} \frac{w(u_2)-w(u_1)}{w(u_3)-w(u_1)} \tag{4}$$

to then rephrase the characterization of Victor as being more averse to dichotomous acts versus safety than Ursula if and only if $f_{\mu,\nu} \leq LE_w$.

Theorem 1 *Victor is more averse to dichotomous acts versus safety than Ursula if and only if* $f_{\mu,\nu} \leq LE_w$.

The lower envelope function. To better understand the lower envelope function, observe that $w_{a,b}(q) = \frac{w(a+q(b-a))-w(a)}{w(b)-w(a)}$ is obtained by *zooming* the graph of w on $[a,b]$ so as to fit exactly in the unit square. The lower envelope function is, as its name suggests, the pointwise infimum of all of these zooms.

$$LE_w(q) = \inf_{a<b} w_{a,b}(q) \tag{5}$$

Proposition 1 *Let LE_w be the lower envelope function of a continuous and strictly increasing function w. Then*

(i) $LE_w(x) \leq x$ for all $x \in [0,1]$, $LE_w(0) = 0$ and $LE_w(1) = 1$.

(ii) $LE_w(x) = x$ for some $x \in (0,1) \Leftrightarrow LE_w(x) = x$ for all $x \in [0,1] \Leftrightarrow w$ is concave.

(iii) LE_w can be either the indicator function of $\{1\}$, the identity function $LE_w(x) \equiv x$ or else is non-decreasing and continuous, with $0 < LE_w(x) < x$ for all $x \in (0,1)$.

(iv) If w is differentiable, defined on a bounded closed interval, LE_w is strictly increasing and self-zooming.

Remark: For concave w the condition $f_{\mu,\nu} \leq LE_w$ simply states that μ must majorize ν ($\forall A, \nu(A) \leq \mu(A)$). Thus, $\{i\}$ if $\mu = \nu$ is a probability measure, this is the Arrow & Pratt characterization of "more risk averse" - concavity of w - and $\{ii\}$ if $\mu \neq \nu$ are probability measures, Victor can't be more risk averse than Ursula. Our main objective is to study increased preference for safety when w is *not* concave.

Proof of Proposition 1: The lower envelope function LE_w is the identity function $LE_w(q) = q$ if and only if w is concave. Indeed, if w is not concave then there is a triple $a < c = a + q(b - a) < b$ such that $(LE_w(q) \leq)\frac{w(c)-w(a)}{w(b)-w(a)} < q$. The other direction holds because $\frac{w(c)-w(a)}{w(b)-w(a)} \geq \frac{c-a}{b-a} = q$ can be made arbitrarily close to q by letting c be a point of continuity of w in the interior of a small enough interval (a, b). This argument also shows that LE_w is bounded above by the identity function. LE_w can certainly be zero everywhere except for $LE_w(1) = 1$. As will now be shown, except for these two extreme cases, LE_w is continuous, positive and below the identity function except at the endpoints. It will also be shown under the further assumption that w is a differentiable function defined on some bounded closed interval, that (i) except for the two extreme cases above, LE_w is strictly increasing, and (ii) LE_w is self-zooming. These two properties may hold more generally, but we don't have a proof.

If w is continuous and strictly increasing but not concave, then there exist $a < c < b$ such that $w(c) < \frac{b-c}{b-a}w(a) + \frac{c-a}{b-a}w(b)$. Furthermore, there exist a maximal $a' \in [a, c)$ and minimal $b' \in (c, b]$ such that $w(a') = \frac{b-a'}{b-a}w(a) + \frac{a'-a}{b-a}w(b)$ and $w(c') = \frac{b-c'}{b-a}w(a) + \frac{c'-a}{b-a}w(b)$ respectively. But then the zoom $w_{[a',c']}$ is strictly below the diagonal, whence so is LE_w. Hence, lower envelope functions are either linear (if and only if w is concave), or strictly below the diagonal except at the endpoints. The other properties stated above follow from the following Lemma 1.

Lemma 1 *For all p and q in $(0, 1)$,*

$$LE_w(pq) \geq LE_w(p)LE_w(q) \ ; \ 1 - LE_w(pq) \leq (1 - LE_w(p))(1 - LE_w(q)) \tag{6}$$

Proof of Lemma 1:

$$\begin{aligned} \frac{w(a + pq(b-a)) - w(a)}{w(b) - w(a)} &= \frac{w(a + pq(b-a)) - w(a)}{w(a + p(b-a)) - w(a)} \frac{w(a + p(b-a)) - w(a)}{w(b) - w(a)} \\ &\geq LE_w(q)\frac{w(a + p(b-a)) - w(a)}{w(b) - w(a)} \geq LE_w(q)LE_w(p) \end{aligned} \tag{7}$$

from which it follows that LE_{pq}, the infimum over $[a, b]$ of the LHS of (7), is also above or on the RHS. A dual inequality for $1 - LE_w$ is proved similarly.

Back to the proof of Proposition 1: Assume that $LE_w(q) > 0$ for some $q \in (0, 1)$. It follows from the first inequality in (6) that for all n, $LE_w(q^n) \geq LE_w(q)^n$. Hence, LE_w is strictly positive throughout $(0, 1)$. The second inequality in (6) implies that $1 - LE_w(q^n) \leq$

$(1-LE_w(q))^n$. Send q to 1 to obtain that for all n, $1-LE_w(1-) \leq (1-LE_w(1-))^n$. Hence, $LE_w(q-) = 1$, or, LE_w is continuous at 1. Now send p to 1 in the first inequality in (6) to see that LE_w is continuous everywhere.

Consider the zoom $w_{a,b}$ of w on an arbitrary interval $[a, b]$. Then (with $LE = LE_w$)

$$\begin{aligned}
\frac{LE(y) - LE(x)}{LE(z) - LE(x)} &\geq \frac{LE(y) - LE(x)}{w_{a,b}(z) - LE(x)} \geq \frac{LE(y) - w_{a,b}(x)}{w_{a,b}(z) - w_{a,b}(x)} \\
&= \frac{w_{a,b}(y) - w_{a,b}(x)}{w_{a,b}(z) - w_{a,b}(x)} - \frac{w_{a,b}(y) - LE(y)}{w_{a,b}(z) - w_{a,b}(x)} \\
&\geq LE(q) - \frac{w_{a,b}(y) - LE(y)}{w_{a,b}(z) - w_{a,b}(x)} = LE(q) - \text{ RATIO} \qquad (8)
\end{aligned}$$

where $q = \frac{y-x}{z-x}$. To infer that LE is self-zooming, we must check that

$$\frac{LE(y) - LE(x)}{LE(z) - LE(x)} \geq LE(q) \qquad (9)$$

Since (8) holds for all $[a, b]$, suppose that the zoom on some $[a, b]$ achieves the envelope value at y. Then the numerator in RATIO is zero, the denominator is positive and we are done. If $LE(y)$ is an infimum over zooms, under the assumption that w is defined on a bounded closed interval, any sequence $[a_i, b_i]$ along which the infimum is achieved, has a convergent subsequence. Clearly, the limit achieves the infimum as a minimum as long as $a = \lim a_i < \lim b_i = b$. In this case, consider any $z \in (0, y)$. Then $LE_w(z) \leq w_{a,b}(z) < w_{a,b}(y) = LE_w(y)$, or, LE_w is strictly increasing on $[0, y]$.

Remark: There is a problem if $a = b$, that is, if $LE(y)$ is achieved infinitesimally, because then the numerator of RATIO may not go to zero faster than the denominator, in which case RATIO would not go to zero. However, if w is differentiable everywhere and $a = b$ then $LE(y)$ can only be y (the diagonal), a case we are not interested in, the concave case.

Simplicity seeking and semi convexity. Victor is a *simplicity seeker* with respect to Ursula if for every act with finite support (say, S) composed of at least three points, there is another act, supported by a strict subset of S, (weakly) preferred by Victor to the original act but such that Ursula is indifferent between the two. The following Theorem 2 introduces a sufficient condition in terms of $f_{\mu,\nu}$ for simplicity seeking. This condition deals with zooms of the capacity transformation function $f_{\mu,\nu}$ of the same nature as in the construction of the lower envelope of the utility transfer function w. It should be clear that if Victor, more averse to dichotomous acts versus safety than Ursula, is also a simplicity seeker with respect to Ursula, then Victor prefers safety more than Ursula under general acts. This is so because (by simplicity seeking) every act with finite support can be replaced by a dichotomous act preferred by Victor and leaving Ursula indifferent, a setup in which the second property takes over.

Consider the "concentric" zooms $Z_{g,q,r}(\Delta) = g_{q-\Delta r, q+\Delta(1-r)}(q)$ of g. g is *semi-convex* if for every $q, r \in (0,1)$, $Z_{g,q,r}(\cdot)$ is minimal at the maximal Δ, the one for which either $q - \Delta r = 0$ or $q + \Delta(1-r) = 1$.

Lemma 2 *Consider throughout the Lemma only "g-functions", i.e., continuous increasing functions on* $[0,1]$ *with fixed points at the endpoints.*

(i) *Every convex g-function is semi-convex.*

(ii) *Every semi-convex g-function is star-shaped at* 0 *and at* 1*, i.e.,* $\frac{g(x)}{x}$ *and* $\frac{1-g(x)}{1-x}$ *are non-decreasing.*

(iii) *A g-function is star-shaped at* 0 *and at* 1 *if and only if it is a pointwise minimum of some family of convex g-functions.*

Proof. To prove (i), observe that if g is convex, $Z_{g,q,r}(\cdot)$ is non-increasing. To prove (ii), observe that by continuity of g, it is enough to show monotonicity at some arbitrary dense set. Suppose the extreme Δ occurs at $q - \Delta r = 0$ and consider the dense set of points q at which g is differentiable. At any such point, send Δ to zero to obtain via $Z_{g,q,r}(0+) = r$ that semi-convexity implies $r \geq \frac{g(q)}{g(\frac{q}{r})}$, or, $\frac{g(q)}{q} \leq \frac{g(\frac{q}{r})}{\frac{q}{r}}$. Similarly if the extreme Δ occurs at $q + \Delta(1-r) = 1$. To prove one direction of (iii), the star-shaped property is satisfied by convexity and is closed under taking minima. To prove the other direction, every g-function that is star-shaped at 0 and 1 is the minimum over q of special convex g-functions, maxima of two linear functions with breakpoint at $(q, g(q))$.

After defining the notion of semi convexity and the property of being a simplicity seeker, we connect the two in Theorem 2, whose proof is the subject matter of Section 5

Theorem 2 *If* $f_{\mu,\nu}$ *is semi-convex, Victor is a simplicity seeker with respect to Ursula, regardless of (increasing)* w.

It will become clear that

Corollary 1 *If $f_{\mu,\nu} \leq g \leq LE_w$ for some semi-convex g, then Victor prefers safety more than Ursula.*

Corollary 2 A necessary and sufficient condition for increasing preference for safety when $f_{\mu,\nu}$ is semi-convex. *Let $f_{\mu,\nu}$ be semi-convex. Then Victor prefers safety more than Ursula if and only if $f_{\mu,\nu} \leq LE_w$.*

Some remarks on RDU. Under risk, there is a probability measure P. RDU can be viewed as CEU with $\mu = g(P)$, $\nu = h(P)$ so that $f_{\mu,\nu}(\cdot) = h(g^{-1}(\cdot))$. Victor, a RDU DM characterized by (v, h), is dichotomous weakly risk averse (prefers safety to dichotomous acts under risk) if and only if he is weakly more dichotomous risk averse than risk neutral Ursula (u and g linear), i.e., if and only if $h \leq LE_v$.

Since lower envelope functions are bounded from above by the identity function, a necessary condition for dichotomous weak risk aversion (thus, for weak risk aversion) in the RDU model is that the DM be *pessimistic*, i.e., $f(q) \leq q$, $\forall q \in [0, 1]$ or, the probability transformation function must be below the diagonal. As shown above, the lower envelope function LE_v is identically equal to the diagonal if and only if v is concave, and this illustrates the well known result by Quiggin [24, 25] that for concave utility on wealth, pessimism and weak risk aversion are equivalent. The probability transformation function f is a functional measure of *pessimism* and the lower envelope function LE_v is a functional measure of *greediness* or non-concavity. Dichotomous risk aversion is characterized by pointwise inequality between the two, much in the spirit of the comparison of the scalar index of pessimism of f and the scalar index of greediness of v introduced by Chateauneuf, Cohen & Meilijson [9], that characterize *monotone* risk aversion.

Does dichotomous weak increased preference for safety imply increased preference for safety? We conjectured that the self-zooming property implies semi convexity. If this were true, by Proposition 1(v) and Corollary 1, the inequality $f_{\mu,\nu} \leq LE_w$ would have fully characterized increased preference for safety, at least for differentiable w defined on a compact interval. Extensive numerical work based on millions of randomly simulated examples failed to produce a counterexample, but neither could we prove the conjecture. This question was settled once a connection was made to Dubins & Savage gambling theory and the remarkable fact that a function g is self-zooming if and only if there is a subfair casino with fixed goal for which g is the "casino function", or optimal probability of winning, as a function of initial wealth. As it turns out, Red & Black casino functions provide counterexamples, not only to the technical question of whether the self-zooming property implies semi convexity, but also to the fundamental question as to whether the inequality $f_{\mu,\nu} \leq LE_w$ that characterizes dichotomous weak increased preference for safety, also characterizes weak increased preference for safety.

Before proving Theorem 2, we digress to gambling theory and exhibit a counterexample to the more general conjecture.

3 Some background on Dubins & Savage [15] casino theory

Casino. A *primitive casino* Γ_r with (loss) rate r specifies the collection $\Gamma_r(x)$ of dichotomous gambles available at every move to a player with wealth $x \in (0,1)$. These gambles permit the player to choose the *stake* $\alpha \in [0, \frac{x}{r}]$. On this move, if initial wealth is x, final wealth is either $x - r\alpha$ or $x + (1-r)\alpha$. Thus, if the gamble is fair (subfair), the probability of winning $g(r)$ is r (strictly less than r). The primitive casino is fully specified by the pair $(r, g(r))$. For our purposes, a *casino* Γ is a union (over r) of primitive casinos, specified by the *rate function* g, assumed herein to be continuous, strictly increasing and strictly below the identity function except for $g(0) = 0$ and $g(1) = 1$. In a primitive casino, the player can choose the stake α. In a casino, the player can choose the stake α and the "hole" r. In a *roulette table*, the player can choose to play simultaneously a number of (stake, hole) pairs.

A move (α, r) is *bold* at $x \leq \frac{r}{r+1}$ if $\alpha = \frac{x}{r}$ and at $x \geq \frac{r}{r+1}$ if $\alpha = \frac{1-x}{1-r}$, in the sense that at least one of the final wealth levels is 0 or 1. The strategy is bold if it only applies bold moves.

Optimality and the self-zooming property. Let $U_g(y)$ be the optimal probability of reaching the goal from initial wealth $y \in (0,1)$, and let $0 < x < y < z < 1$ with $\frac{y-x}{z-x} = q$. By optimality, an interim goal of reaching z versus x should be detrimental to chance of winning. Since the optimal probability of reaching z rather than x is $U_g(q)$, the following inequality holds

$$U_g(y) \geq U_g(q)U_g(z) + (1 - U_g(q))U_g(x) \tag{10}$$

in other words (see (9)), U_g satisfies the self-zooming property

$$U_g(q) \leq \frac{U_g(y) - U_g(x)}{U_g(z) - U_g(x)} \tag{11}$$

By Theorem 4.2.1. in Dubins & Savage [15] (see also expression 4.2.8), self-zooming functions are precisely those that can be obtained as the optimal probability U_g of reaching the goal in subfair casinos.

4 The counterexample

Let U_g be the optimal probability of reaching the goal as a function of initial wealth, as defined in the previous section. This function is thus self-zooming. Suppose that Ursula is characterized by the pair (u, μ) and Victor by (v, ν), where $v = U_g(u)$ and $\nu = U_g(\mu)$. In other words, U_g plays the double role of utility transfer function w and capacity transformation function f. Since $LE_w \geq f$ (because $LE_w = U_g = f$), Victor is dichotomous weakly more uncertainty averse than Ursula.

We will now see that for U_g of Red & Black, a specific casino, there is a 3-valued act that Victor prefers to safety preferred by Ursula.

In casino language, Red & Black (see [15]) is the primitive casino with $r = \frac{1}{2}$ and some $g(\frac{1}{2}) = \omega \in (0, \frac{1}{2})$. The optimal probability (reached under bold play, the seminal result of Dubins & Savage [15] for primitive casinos) of ever reaching the goal from $\frac{1}{2}$ is $U(\frac{1}{2}) = \omega$, then $U(\frac{1}{4}) = U(\frac{1}{2})^2 = \omega^2$ and $U(\frac{3}{4}) = U(\frac{1}{2}) + (1 - U(\frac{1}{2}))U(\frac{1}{2}) = \omega + (1 - \omega)\omega$. More generally, $U(\frac{k}{2^n})$ can be recursively expressed in terms of the corresponding probabilities $U(\frac{l}{2^{n-1}})$. There is a unique monotone extension of U from the binary rationals in the unit interval to the unit interval, so U is well defined.

The following table lists, for the case $\omega = 0.3$, a few points and their U values as well as their roles in building a distribution with values $x1, x2, x3$ and respective probabilities $1 - q1, q1 - q2, q2$.

point * 2048	U	role
0	0	
2	$5.9049 * 10^{-6}$	x_1
1024	0.3000	ω
1366	0.3800	x_2
1537	0.5100	q_2
1580	0.5154	$> E[X]$
1600	0.5232	q_1
2048	1	x_3

Let X be the 3-valued act with distribution $(\frac{2}{2048}, 0.4768\ ;\ \frac{1366}{2048}, 0.0132\ ; 1, 0.51)$. The expectation of this distribution is $E[X] = 0.7712 = 1579.5/2048 < 1580/2048$. Hence, the utility at the expectation is less than $U(1580/2048) = 0.5154$, but $RDU(X)$ is computed to be 0.5182. In other words, Victor, dichotomous weakly more uncertainty averse than EU Ursula, is **not** weakly more uncertainty averse than Ursula.

This exhibits an example of a self-zooming function that is not semi-convex. We leave as an open problem the investigation of these functions. Attempts at bumping into one in the street failed miserably:

Randomly sampled self-zooming functions. Continuous increasing functions u with $u(0) = 0$ and $u(1) = 1$ may be randomly sampled by first sampling $u(\frac{1}{2})$ uniformly in the unit interval, then sampling $u(\frac{1}{4})$ uniformly in $(0, u(\frac{1}{2}))$ and $u(\frac{3}{4})$ uniformly in $(u(\frac{1}{2}), 1)$, etc., to the desired resolution, and then interpolating the definition piece-wise linearly. Instead of using binary rational knots, it is also possible to use randomly sampled knots, as investigated by Dubins & Freedman [14]. Once such a random piece-wise linear function has been determined, it is possibly to compute accurately its lower envelope function, to obtain a randomly sampled self-zooming function. This operation was repeated thousands upon thousands of times with up to 128 knots, never ever producing a non semi-convex function!

Furthermore, the function U listed above failed to produce a counterexample too, when values of X and their probabilities were restricted to be multiples of 1/256. A Dynamic Programming program designed to construct an optimal distribution for Victor exceeded the computer's memory when attempting multiples of 1/512...

As a consequence, it seems that "most" cases of dichotomous weak increased uncertainty aversion display general weak uncertainty aversion, but we have been unable to formalize this feeling.

5 Proofs and further analysis

Theorem 2 will be proved first in RDU language. The proof will then be extended to cover CEU. Consider a random variable X supported by $x_1 < x_2 < \cdots < x_n$ with $n \geq 3$ and $P(X > x_i) = q_i$; $1 \leq i < n$. It will be shown that there is another distribution, with the same mean and support as X, except for missing one of the three leftmost atoms, leaving all other atoms intact in value and probability, preferred by the DM. Analysis will be split according to whether $E[X|X \leq x_3]$ is to the left or to the right of x_2.

Case 1. $E[X|X \leq x_3] \geq x_2$. The removal of the atom x_1 and the assignment $p_1 = q_2 - (1-q_1)\frac{x_2-x_1}{x_3-x_2}$ entail the comparison

$$\begin{aligned} & (1-f(q_1))u(x_1) + (f(q_1)-f(q_2))u(x_2) + (f(q_2)-f(q_3)u(x_3)) + \cdots \\ \leq \quad & (1-f(p_1))u(x_2) + (f(p_1)-f(q_3))u(x_3) + \cdots \end{aligned} \tag{12}$$

or

$$\frac{u(x_2)-u(x_1)}{u(x_3)-u(x_1)} \geq \frac{f(q_2)-f(p_1)}{1-f(q_1)+f(q_2)-f(p_1)} \tag{13}$$

between two (objective) distributions with the same mean. Similarly, the removal of the atom x_2 and the assignment $p_2 = q_1\frac{x_2-x_1}{x_3-x_1} + q_2(1-\frac{x_2-x_1}{x_3-x_1})$ entail, as before, the comparison

$$\begin{aligned} & (1-f(q_1))u(x_1) + (f(q_1)-f(q_2))u(x_2) + (f(q_2)-f(q_3)u(x_3)) + \cdots \\ \leq \quad & (1-f(p_2))u(x_1) + (f(p_2)-f(q_3))u(x_3) + \cdots \end{aligned} \tag{14}$$

or

$$\frac{u(x_2)-u(x_1)}{u(x_3)-u(x_1)} \leq \frac{f(p_2)-f(q_2)}{f(q_1)-f(q_2)} \tag{15}$$

between distributions with equal means.

At least one of (13) and (15) is satisfied if and only if

$$\frac{f(q_2)-f(p_1)}{1-f(q_1)+f(q_2)-f(p_1)} \leq \frac{f(p_2)-f(q_2)}{f(q_1)-f(q_2)} \tag{16}$$

In other words, if and only if

$$\frac{f(p_2)-f(p_1)}{1-f(p_1)} \leq \frac{f(p_2)-f(q_2)}{f(q_1)-f(q_2)} \tag{17}$$

Inequality (17) is clearly satisfied when f is semi-convex: The LHS and RHS of (17) are zooms of f around p_2 with objective rate

$$r = \frac{p_2-p_1}{1-p_1} = \frac{p_2-q_2}{q_1-q_2} = \frac{x_2-x_1}{x_3-x_1} \tag{18}$$

As such, the LHS of (17) is a maximal zoom-out of the RHS: fix p_2, fix r and let q_1 be defined by (18) as a function of q_2. Then, if f is semi-convex, the LHS of (17) is the minimum of the RHS.

Case 2. $E[X|X \leq x_3] \leq x_2$. The removal of the atom x_3 and the assignment $p_3 = q_1 + q_2\frac{x_3-x_2}{x_2-x_1}$ preserves means and entails the comparison

$$\begin{aligned} & (1-f(q_1))u(x_1) + (f(q_1)-f(q_2))u(x_2) + (f(q_2)-f(q_3)u(x_3)) + \cdots \\ \leq \quad & (1-f(p_3))u(x_1) + (f(p_3)-f(q_3))u(x_2) + (f(q_3)-f(q_3))u(x_3) + \cdots \end{aligned} \tag{19}$$

or

$$(f(p_3) - f(q_1))(u(x_1) - u(x_2)) + f(q_2)(u(x_3) - u(x_2)) \leq f(q_3)(u(x_3) - u(x_2)) \tag{20}$$

Inequality (20) is satisfied for all $q_3 \geq 0$ if and only if it is satisfied for the most stringent value $q_3 = 0$, in which case the inequality becomes

$$\frac{u(x_2) - u(x_1)}{u(x_3) - u(x_1)} \geq \frac{f(q_2)}{f(q_2) + f(p_3) - f(q_1)} \tag{21}$$

As in case 1, at least one of (15) and (21) is satisfied if and only if

$$\frac{f(q_2)}{f(q_2) + f(p_3) - f(q_1)} \leq \frac{f(p_2) - f(q_2)}{f(q_1) - f(q_2)} \tag{22}$$

In other words, if and only if

$$\frac{f(p_2)}{f(p_3)} \leq \frac{f(p_2) - f(q_2)}{f(q_1) - f(q_2)} \tag{23}$$

This inequality is of the same nature as (17), satisfied by semi-convex f for the same reason.

The proof of Theorem 2 under RDU is essentially finished. We have seen that DMs with semi-convex f always prefer some equal-mean distribution missing one of the three leftmost atoms. The recursive application of this idea will end up with a preferred dichotomous equal-mean distribution.

In CEU, with $f_{\mu,\nu} = f$, express the Choquet index for Victor in terms of Ursula's utility on wealth $u_i = U(x_i)$ and capacity $q_i = \mu(A_i)$ as

$$\begin{aligned} V(X) &= w(u_1) + \nu(A_1)(w(u_2) - w(u_1)) + \nu(A_2)(w(u_3) - w(u_2) + \cdots \\ &\leq w(u_1) + f(q_1)(w(u_2) - w(u_1)) + f(q_2)(w(u_3) - w(u_2) + \cdots \end{aligned} \tag{24}$$

Now apply the RDU result verbatim, with Ursula's Choquet index $U(X)$ playing the role of expectation.

To prove Corollary 1 in RDU language, observe that by monotonicity of RDU with respect to f, $RDU_{u,f}(X) \leq RDU_{u,g}(X)$. But g is semi-convex, so there is a dichotomous act Y with $E[Y] = E[X]$ and $RDU_{u,g}(X) \leq RDU_{u,g}(Y)$. Since (u, g) displays dichotomous weak risk aversion, $RDU_{u,g}(Y) \leq u(E[Y])$ and $RDU_{u,f}(X) \leq u(E[X])$.

Corollary 2, the most conclusive result of this study, is an immediate consequence.

6 Some examples

Example 1: a pair (w, f) displaying weak but not monotone risk aversion. If $w(x) = x^2$ on $[0, 1]$ and $f(p) = p^2$, the RDU DM is weakly risk averse because w is self-zooming and identical to (convex) f.

The probability transformation function $f(p) = p^2$. Its scalar index of pessimism is $P_f = \inf_p \frac{1-p^2}{p^2}\frac{p}{1-p} = \inf_p \frac{1+p}{p} = 2$. Thus, a DM with this probability transformation function is monotone risk averse if and only if the marginal utility of one unit of money is never more than twice what it would have been had the DM been poorer.

The utility function $w(x) = x^2$ on $[0, \infty)$. Since this utility function is convex, the only slopes to compare for the computation of the scalar index of greediness are the derivatives at the endpoints of the interval of reference. On $[0, \infty)$ or even on $[0, a]$ for positive but finite a, this DM is so greedy on account of utility that the only possibility for being monotone risk averse is infinite pessimism, given exclusively by the probability transformation function with $f(p) = 0$ for all $p < 1$. Restriction to random variables supported by an interval with positive endpoints will reduce greediness and make the DM more averse to risk. However, as long as $G_w > 2$, this weakly risk averse DM is not monotone risk averse.

Example 2: Choquet Expected Utility. Consider three states of nature $\{1, 2, 3\}$. Let Ursula be characterized by the pair (u, μ), where $u(x) = \sqrt{x}$ and μ is the uniform distribution on $\{1, 2, 3\}$. Let Victor be characterized by the pair (v, ν) given by $v(x) = x$ and a Choquet capacity ν that assigns mass $\frac{1}{9}$ to every singleton and $\frac{4}{9}$ to every two-point set. We can see that $w(x) = x^2$ and f can be taken as $f(x) = x^2$. Hence, from the previous example, (i) Victor is weakly more uncertainty averse than Ursula, even if the utility function of Ursula is concave in that of Victor, and (ii) Victor is *not* monotone more uncertainty averse than Ursula.

7 Concluding remark

In laboratory experiments it is easy to design questions that discriminate between attitudes towards risk and attitudes towards ambiguity. This is no more the case in surveys based on questions of everyday life choices. The notion of "more preference for safety" applied in this paper should allow a typology that classifies individuals into different categories according to their degree of preference for safety, i.e., their degree of weak uncertainty aversion.

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